MAT-51316 Partial Differential Equations
Exam 20.5.2011

No books, no notes, no calculator. Write solutions in English or in Finnish. There are formulas on the back of this page.
Ei kirjoja, ei muistiinpanoja, ei laskinta. Kirjoita ratkaisut englanniksi tai suomeksi. Kääntöpuolella on kaavoja.

1. (a) Explain Fick's law of diffusion $\phi = -ku_x$.
   (b) Show that information in the initial condition of a one-dimensional heat equation $u_t = ku_{xx}$ for $x \in \mathbb{R}$ and $t > 0$ propagates with infinite speed.

2. (a) Consider a one dimensional wave equation $u_{tt} = \frac{1}{4}u_{xx}$ on the half-line $x > 0$ subject to the boundary condition $u(0,t) = 0$ and the initial conditions $u_t(x,0) = 0$ and
$$u(x,0) = \begin{cases} 1 - |x-2| & \text{if } |x-2| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$
   Plot the solution $u(x,t)$ at $t = 3$ and at $t = 6$.
   (b) What is the *domain of dependence* of the solution of the problem in part (a) at $x = 1$, $t = 5$?

3. Find the coefficients of the Fourier series
$$s(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(2nx) + b_n \sin(2nx)$$
   for the function $f(x) = |\sin x|$. To what function does $s(x)$ converge to (pointwise)?

4. Find the steady-state temperature in a long cylinder of radius a if the outside temperature is maintained at $u = A\sin^2\theta$. What is the temperature in the centre?

5. Show that if $u'' + cu = 0$ on (a,b) with $u(a) = h_0$, $u'(b) = h_1$ then $u(\xi) = G_x(a,\xi)h_0 + G(b,\xi)h_1$ for $\xi \in (a,b)$, where $G(x,\xi)$ is the Green's function for this differential operator and these boundary conditions. Hint: $G(x,\xi)$ satisfies the following conditions:
   - $G_x(x^+,x) - G_x(x^-,x) = -1$
   - $G_{xx} + cG = 0 \quad (x \neq \xi)$
   - $G(a,\xi) = 0,\ G_x(b,\xi) = 0$

kaavoja / formulas

$$u_t + (cu)_x = f, \quad u_{tt} - c^2 u_{xx} = f, \quad u_t - (ku_x)_x = f, \quad \Delta u = u_{xx} + u_{yy} + u_{zz}$$

$$u(x,t) = \tfrac{1}{2}\phi(x+ct) + \tfrac{1}{2}\phi(x-ct) + \tfrac{1}{2c}\int_{x-ct}^{x+ct} \psi(\xi)\,\mathrm{d}\xi$$

$$u(x,t) = \int_{-\infty}^{\infty} S(x-\xi,t)\phi(\xi)\,\mathrm{d}\xi, \quad S(x,t) = \tfrac{1}{\sqrt{4\pi kt}} e^{-x^2/(4kt)}$$

$$\mathrm{erf}(x) = \tfrac{2}{\sqrt{\pi}}\int_0^x e^{-p^2}\,\mathrm{d}p, \quad \mathrm{erf}(\infty) = 1$$

$\mathbf{S}(t)\phi$ ratkaisee/solves " $u_t + \mathbf{A}u = 0,\ u(0) = \phi$ "

$\Rightarrow \int_0^t \mathbf{S}(t-\tau)f(\tau)\,\mathrm{d}\tau$ ratkaisee/solves " $u_t + \mathbf{A}u = f,\ u(0) = 0$ "

$$v(x,t) = \int_0^\infty \underbrace{[S(x-\xi,t) - S(x+\xi,t)]}_{S_{\mathrm{halfline}}(x,\xi,t)} \phi(\xi)\,\mathrm{d}\xi$$

$$(\kappa X_m' X_n - \kappa X_n' X_m)' = (\lambda_n - \lambda_m)\mu X_m X_n$$

$$u' + \lambda u = 0 \Leftrightarrow u(x) = Ae^{-\lambda x}$$

$$u'' + \beta^2 u = 0 \Leftrightarrow u(x) = A\cos(\beta x) + B\sin(\beta x)$$

$$u'' - \beta^2 u = 0 \Leftrightarrow u(x) = A\cosh(\beta x) + B\sinh(\beta x)$$

$$\cos(u+v) = \cos(u)\cos(v) - \sin(u)\sin(v), \quad \sin(u+v) = \sin(u)\cos(v) + \cos(u)\sin(v)$$

$$f(x) = \sum_{n\geq 1}\left(\frac{2}{l}\int_0^l f(\xi)\sin(n\pi\xi/l)\,\mathrm{d}\xi\right)\sin(n\pi x/l)$$

$$f(x) = \frac{1}{2l}\int_{-l}^{l} f(\xi)\,\mathrm{d}\xi + \sum_{n\geq 1}\left(\frac{1}{l}\int_{-l}^{l} f(\xi)\cos(n\pi\xi/l)\,\mathrm{d}\xi\right)\cos(n\pi x/l)$$
$$+ \sum_{n\geq 1}\left(\frac{1}{l}\int_{-l}^{l} f(\xi)\sin(n\pi\xi/l)\,\mathrm{d}\xi\right)\sin(n\pi x/l)$$

$$\int_D \nabla\cdot\mathbf{f}\,\mathrm{d}V = \int_{\partial D}\mathbf{f}\cdot\mathbf{n}\,\mathrm{d}A, \quad \int_{\partial D}(v\nabla u)\cdot\mathbf{n}\,\mathrm{d}A = \int_D (\nabla v\cdot\nabla u + v\Delta u)\,\mathrm{d}V$$

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi}\frac{(a^2-r^2)h(\phi)}{a^2 - 2ar\cos(\theta-\phi) + r^2}\,d\phi$$

$$\Delta u = \tfrac{1}{r^2}\left(r^2 u_r\right)_r + \tfrac{1}{r^2\sin\theta}\left(u_\theta\sin\theta\right)_\theta + \tfrac{1}{r^2\sin^2\theta}u_{\phi\phi}$$

$$\Delta u = \tfrac{1}{R}\left((Ru_R)_R + \tfrac{1}{R}(u_\theta)_\theta + (Ru_z)_z\right)$$