MAT-51316 Partial Differential Equations
Midterm exam 8.3.2011

No books, no notes, no calculator. Write solutions in English or in Finnish.
Ei kirjoja, ei muistiinpanoja, ei laskinta. Kirjoita ratkaisut englanniksi tai suomeksi. Kääntöpuolella on kaavoja.

1. Solve the equation $u_t + 2u_x + xu_x + u = 0$ with the initial condition $u(x,0) =$ e[illegible], and sketch some characteristic cures. What does the equation model?
   $-(x-1)^2$

2. Derive the Robin boundary condition $ku_x(l,t) + \kappa u(l,t) = \kappa g(t)$ for the one-dimensional heat equation $u_t - (ku_x)_x = f$.

3. Show that the eigenvalues of the eigenfunction ODE

   $$X''(x) + \lambda\mu(x)X(x) = 0 \quad (0 < x < l)$$

   with $\mu(x) > 0$ and boundary conditions

   $$X'(0) = \alpha X(0), \quad X'(l) = 0$$

   with $\alpha > 0$, are real and positive.

kaavoja / formulas

$u_t + (cu)_x = f, \quad u_{tt} - c^2 u_{xx} = f, \quad u_t - (ku_x)_x = f, \quad \Delta u = u_{xx} + u_{yy} + u_{zz}$

$u(x,t) = \frac{1}{2}\phi(x+ct) + \frac{1}{2}\phi(x-ct) + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(\xi)\,\mathrm{d}\xi$

$u(x,t) = \int_{-\infty}^{\infty} S(x-\xi,t)\phi(\xi)\,\mathrm{d}\xi, \quad S(x,t) = \frac{1}{\sqrt{4\pi kt}}e^{-x^2/(4kt)}$

$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-p^2}\,\mathrm{d}p, \quad \mathrm{erf}(\infty) = 1$

$\mathbf{S}(t)\phi$ ratkaisee/solves " $u_t + \mathbf{A}u = 0,\ u(0) = \phi$ "

$\Rightarrow \int_0^t \mathbf{S}(t-\tau)f(\tau)\,\mathrm{d}\tau$ ratkaisee/solves " $u_t + \mathbf{A}u = f,\ u(0) = 0$ "

$v(x,t) = \int_0^\infty \underbrace{[S(x-\xi,t) - S(x+\xi,t)]}_{S_{\text{halfline}}(x,\xi,t)}\phi(\xi)\,\mathrm{d}\xi$

$(\kappa X_m' X_n - \kappa X_n' X_m)' = (\lambda_n - \lambda_m)\mu X_m X_n$

$u' + \lambda u = 0 \Leftrightarrow u(x) = Ae^{-\lambda x}$

$u'' + \beta^2 u = 0 \Leftrightarrow u(x) = A\cos(\beta x) + B\sin(\beta x)$

$u'' - \beta^2 u = 0 \Leftrightarrow u(x) = A\cosh(\beta x) + B\sinh(\beta x)$

$\cos(u+v) = \cos(u)\cos(v) - \sin(u)\sin(v), \quad \sin(u+v) = \sin(u)\cos(v) + \cos(u)\sin(v)$

$$f(x) = \sum_{n\geq 1}\left(\frac{2}{l}\int_0^l f(\xi)\sin(n\pi\xi/l)\,\mathrm{d}\xi\right)\sin(n\pi x/l)$$

$$f(x) = \frac{1}{2l}\int_{-l}^{l} f(\xi)\,\mathrm{d}\xi + \sum_{n\geq 1}\left(\frac{1}{l}\int_{-l}^{l} f(\xi)\cos(n\pi\xi/l)\,\mathrm{d}\xi\right)\cos(n\pi x/l)$$
$$+ \sum_{n\geq 1}\left(\frac{1}{l}\int_{-l}^{l} f(\xi)\sin(n\pi\xi/l)\,\mathrm{d}\xi\right)\sin(n\pi x/l)$$

$$\int_D \nabla\cdot\mathbf{f}\,\mathrm{d}V = \int_{\partial D}\mathbf{f}\cdot\mathbf{n}\,\mathrm{d}A, \quad \int_{\partial D}(v\nabla u)\cdot\mathbf{n}\,\mathrm{d}A = \int_D (\nabla v\cdot\nabla u + v\Delta u)\,\mathrm{d}V$$

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi}\frac{(a^2 - r^2)h(\phi)}{a^2 - 2ar\cos(\theta-\phi) + r^2}\,d\phi$$

$\Delta u = \frac{1}{r^2}\left(r^2 u_r\right)_r + \frac{1}{r^2\sin\theta}\left(u_\theta\sin\theta\right)_\theta + \frac{1}{r^2\sin^2\theta}u_{\phi\phi}$

$\Delta u = \frac{1}{R}\left((Ru_R)_R + \frac{1}{R}(u_\theta)_\theta + (Ru_z)_z\right)$